NOTES ON MATRIX THEORY—IX

Richard Bellman

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The RAND Corporation
1700 MAIN ST. • SANTA MONICA • CALIFORNIA

SUMMARY

Using a generalization of an identity of Siegel, a concavity theorem is established for power products of the form $|X_1|^{a_1} |X_2|^{a_2} \cdots |X_n|^{a_n}$, where $|X_k| = |x_{ij}|$, $i,j = 1, 2, \ldots, k$.

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by

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§1. Introduction

In a recent note, [1], we showed that the inequality

$$(1) \qquad |\lambda A + (1-\lambda)B| \geq |A|^{\lambda} \, |B|^{1-\lambda},$$

valid for positive definite matrices A and B, for $0 \leq \lambda \leq 1$, was a simple consequence of Holder's inequality and the identity

$$(2) \qquad \int_0^{\infty} e^{-(x,Cx)} \prod_1^n dx_i = \sqrt{\pi}^{\,n} / |C|^{1/2},$$

for C a positive definite matrix of order n.

In this note we wish to use a more recondite identity, a generalization of an integral of Ingham and Siegel, due to A. Selberg, to derive an extensive generalization of (1), namely

Theorem. Let A and B be two positive definite matrices of order n, and let $C = \lambda A + (1-\lambda)B$, for $0 \leq \lambda \leq 1$. For each $j = 1, 2, \ldots, n$, let $A^{(j)}$ denote the principal submatrix of A obtained by deleting the first $(j-1)$ rows and columns, (in particular, $A^{(1)} = A$). Let $B^{(j)}$, $C^{(j)}$, have similar meanings. If $k_1, k_2, \ldots, k_n$ are n real numbers such that

(3) $$\sum_{i=1}^{j} k_i \geq 0, \qquad j = 1, 2, \ldots, n,$$

then

(4) $$\prod_{j=1}^{n} |C^{(j)}|^{k_j} \geq \prod_{j=1}^{n} |A^{(j)}|^{\lambda k_j} \, |B^{(j)}|^{(1-\lambda)k_j} .$$

The above sharp form of the inequality is due to a referee. We shall first present his proof below, and then the proof of a particular case, derived from the identity mentioned above.

§2. Proof of Theorem

According to Bergstrom's inequality, [2], or a minimum theorem due to Fan [4], we have for $j = 1, 2, \ldots, n-1$,

(1) $$\frac{|C^{(j)}|}{|C^{(j+1)}|} \geq \frac{|\lambda A^{(j)}|}{|\lambda A^{(j+1)}|} + \frac{|(1-\lambda)B^{(j)}|}{|(1-\lambda)B^{(j+1)}|}$$

$$= \lambda \frac{|A^{(j)}|}{|A^{(j+1)}|} + (1-\lambda) \frac{|B^{(j)}|}{|B^{(j+1)}|}$$

$$\geq \left(\frac{|A^{(j)}|}{|A^{(j+1)}|}\right)^{\lambda} \left(\frac{|B^{(j)}|}{|B^{(j+1)}|}\right)^{(1-\lambda)} .$$

The desired inequality follows upon writing

$$(2) \qquad \prod_{j=1}^{n} |C^{(j)}|^{k_j} = \left(\frac{|C^{(1)}|}{|C^{(2)}|}\right)^{k_1} \left(\frac{|C^{(2)}|}{|C^{(3)}|}\right)^{k_1+k_2}$$

$$\cdots \left(\frac{|C^{(n-1)}|}{|C^{(n)}|}\right)^{k_1+k_2+\cdots+k_{n-1}} |C^{(n)}|^{\sum_{i=1}^{n} k_i} ,$$

and using the condition that $\sum_{i=1}^{k_1} k_i \geq 0$, together with the inequality above.

§3. Partial Proof

It was shown by Siegel, [6], p. 44, that the following generalization of the gamma function integral exists:

$$(1) \qquad \int_{X>0} e^{-tr(XY)} |X|^{s-\left(\frac{n-1}{2}\right)} \prod_{i\leq j} dx_{ij} = a_n |Y|^{-s} .$$

Here X and Y are symmetric matrices of order n, with Y positive definite, and the integration is extended over the region of x_{ij}-space in which X is positive definite.

The constant a_n is given by

$$(2) \qquad a_n = \pi^{\frac{n(n-1)}{4}} \Gamma(s)\Gamma(s - \tfrac{1}{2}) \cdots \Gamma(s - (\tfrac{n-1}{2})) .$$

The integral converges for $Re(s) > (\frac{n-1}{2})$, and equals the right-hand side.

It was pointed out to the author by A. Selberg that an extension of Siegel's integral exists, namely

$$(3) \qquad \int_{X>0} e^{-tr(XY)} |X^{(1)}|^{\sum_{i=1}^{n} k_i - (\frac{n+1}{2})} |X^{(2)}|^{-k_1} \cdots |X^{(n)}|^{-k_{n-1}} \prod dx_{ij}$$

$$= b_n |Y_n|^{-k_n} |Y_{n-1}|^{-k_{n-1}} \cdots |Y_1|^{-k_1} ,$$

where $X^{(j)}$ is as above, $Y_j = (y_{ij})$, $1 \leq i,j \leq k$, and

$$(4) \qquad b_n = \pi^{\frac{n(n-1)}{4}} \Gamma(k_n)\Gamma(k_n + k_{n-1} - \tfrac{1}{2}) \cdots \Gamma(\sum_{i=1}^{n} k_i - (\tfrac{n+1}{2})) .$$

The integral exists and has the stated value provided that each of the expressions k_n, $k_n + k_{n-1} - \frac{1}{2}$, ... $\sum_{i=1}^{n} k_i - (n + \frac{1}{2})$ is positive. Once we have a representation for $|Y_n|^{-k_n} |Y_{n-1}|^{-k_{n-1}} \cdots |Y_1|^{-k_1} = \Psi(Y)$ in the form

$$(5) \qquad \Psi(Y) = \int_{X>0} \phi(X) e^{-tr(XY)} \prod_{i \leq j} dx_{ij} ,$$

with $\phi \geq 0$, the proof proceeds as in [1].

A proof of (3) and an analogous extension of an integral of Ingham [5] equivalent to Siegel's may be found in [3], together with some applications.

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